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# Structure constants of $\mathfrak{shs}[\lambda]$ : the deformed-oscillator point of view

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## Abstract

We derive and spell out the structure constants of the  $\mathbb{Z}_2$ -graded algebra  $\mathfrak{shs}[\lambda]$  by using deformed-oscillators techniques in  $Aq(2; \nu)$ , the universal enveloping algebra of the Wigner-deformed Heisenberg algebra in 2 dimensions. The use of Weyl ordering of the deformed oscillators is made throughout the paper, via the symbols of the operators and the corresponding associative, non-commutative star product. The deformed oscillator construction was used by Vasiliev in order to construct the higher spin algebras in three spacetime dimensions. We derive an expression for the structure constants of  $\mathfrak{shs}[\lambda]$  and show that they must obey a recurrence relation as a consequence of the associativity of the star product. We solve this condition and show that the  $\mathfrak{hs}[\lambda]$  structure constants are given by those postulated by Pope, Romans and Shen for the Lone Star product.

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## 1 Introduction

Three dimensional spacetime constitutes a particularly interesting testing ground for the study of Higher Spin (HS) theories, as some of the technical difficulties appearing in dimensions 4 and higher are absent due to the topological nature of gauge fields of spin 2 and higher.

In the presence of a negative cosmological constant in 3D and without any coupling between HS fields and matter, a standard action principle for an infinite tower of gauge fields with integer spin  $s \geq 2$  is given [1] by (the difference of) two Chern–Simons actions for the infinite dimensional superalgebras studied by Fradkin and Vasiliev, see [2, 3]. A doubling of the algebra, necessary in order to produce the direct sum in the three-dimensional anti-de Sitter ( $\text{AdS}_3$ ) isometry algebra  $\mathfrak{so}(2, 2) \cong \mathfrak{so}(2, 1) \oplus \mathfrak{so}(2, 1)$ , can be achieved by introducing an outer element  $\psi$  satisfying  $\psi^2 = 1$ . As is customary in this context, one does not always mention this doubling of algebras that is implicitly understood. As shown in [4], these (super)algebras are characterized by a real parameter  $\nu$  and possess a unique nondegenerate supertrace<sup>3</sup>. For critical values  $\nu = -(2\ell + 1)$ ,  $\ell \in \mathbb{N}$ , an ideal appears and can be quotiented out, leaving a finite-dimensional bosonic subalgebra  $\mathfrak{gl}(N)$  where  $N = \ell + 1$ , see e.g. [9] for a recent review and extensions, together with the notations that we adopt here.<sup>4</sup> In the critical case, the  $\mathfrak{sl}(N) \oplus \mathfrak{sl}(N)$ -valued connection describes gauge fields with spin  $s = 2, 3, \dots, N$ , thereby naturally extending the classical reformulation of three-dimensional gravity [10, 11], see [12] for a pedagogical review. The Blencowe construction [1] corresponds to taking  $\nu = 0$ .

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<sup>3</sup>The existence of a one parameter family of higher spin algebras is usually considered to be a special feature of dimension 3, but it is not the case, see [5, 6] on  $\text{AdS}_5$  and [7, 8] in general dimension.

<sup>4</sup>The Chern–Simons construction in [9] unifies HS fields with fractional spin-fields and an internal nonabelian sector.

Focusing on the purely bosonic case, the HS gauge algebras that generalise the one used by Blencowe are denoted  $\mathfrak{hs}[\lambda]$  and defined as follows:

$$\mathbb{C} \oplus \mathfrak{hs}[\lambda] = \frac{\mathcal{U}(\mathfrak{sl}(2, \mathbb{R}))}{\langle \mathcal{C}_2 - \mu \mathbb{1} \rangle}, \quad \mu = \frac{\lambda^2 - 1}{4}, \quad (1.1)$$

where  $\mathcal{U}(\mathfrak{sl}(2, \mathbb{R}))$  is the universal enveloping algebra (UEA) of  $\mathfrak{sl}(2, \mathbb{R})$ ,  $\mathcal{C}_2$  its quadratic Casimir and  $\langle \mathcal{C}_2 - \mu \mathbb{1} \rangle$  the ideal generated by the relation in brackets, i.e. the value of  $\mathcal{C}_2$  is fixed. The relation between the parameters  $\nu$  and  $\lambda$  is given by  $\lambda = \frac{1-\nu}{2}$ . When the value of  $\nu$  is non critical, the Chern–Simons model describes an infinite tower of non-propagating but interacting HS fields of spin  $s = 2, 3, \dots$

The Prokushkin–Vasiliev (PV) equations [13] precisely describe an infinite tower of higher spin gauge fields, coupled with two (or one) complex scalar field. As in the Chern–Simons formulation, the PV equations are based on the gauging of  $\mathfrak{hs}[\lambda]$ . The analysis of the spectrum of the PV equations is not straightforward however, due to the presence of extra kleinians and twisted sectors; see e.g. [14, 15] for recent studies and in particular [14] for a thorough treatment of the twisted sector. Even though in the Blencowe formulation one benefits from the standard technology from Chern–Simons actions, matter coupling is not possible, or at least it is not yet known how to implement it. For some works in that direction, see [16]. Note, however, that it is possible to give an action for matter-coupled 3D HS fields, see [17]. There, it was shown how the Chern–Simons description emerges upon consistent truncation and at the expense of losing the matter sector. In [17, 15] it was also shown how to further truncate the spectrum to only one *real* scalar field, of mass  $m^2 = -1 + \lambda^2$ . This last feature may be interesting in the context of the work [18] and after the more recent developments where it was understood that one of the two scalar fields should correspond to non-perturbative degrees of freedom, see the review [19].

The PV model has witnessed a surge of interest from the fact that the bulk theory governed by the PV equations has been conjectured by Gaberdiel and Gopakumar [20, 19] to be dual to  $\mathcal{W}_\infty[\lambda]$  minimal model CFTs. More precisely, the CFT considered is a Wess–Zumino–Witten coset model:

$$\frac{\mathrm{SU}(N)_k \otimes \mathrm{SU}(N)_1}{\mathrm{SU}(N)_{k+1}} \quad (1.2)$$

in the t’Hooft limit

$$N, k \rightarrow \infty, \quad \text{with } \lambda = \frac{N}{N+k} \text{ fixed.} \quad (1.3)$$

The t’Hooft parameter  $\lambda$  is to be identified with the parameter fixing the  $\mathfrak{sl}(2, \mathbb{R})$  quadratic Casimir value in the definition of  $\mathfrak{hs}[\lambda]$ . Promising results in this holographic context have been obtained [21, 22, 23], in favour of HS theories seen as a tensionless limit of String Theory, see refs. therein.

A realisation of  $\mathfrak{hs}[\lambda]$  was given in PV’s original paper [13], elaborating on the previous work of Vasiliev [4] using the so-called *Wigner-deformed oscillators* [24, 25], see also [26, 27], while [8] gives

a matrix-valued realisation of the deformed oscillators. They constitute a deformation of the usual oscillators as they verify the following commutation relation:

$$[\hat{q}_\alpha, \hat{q}_\beta] = 2i \epsilon_{\alpha\beta} (1 + \hat{k}\nu), \quad \{\hat{q}_\alpha, \hat{k}\} = 0, \quad \nu \in \mathbb{R} \quad (1.4)$$

in terms of a real spinor  $\hat{q}_\alpha$  ( $\alpha = 1, 2$ ) of  $\mathfrak{sl}(2, \mathbb{R})$  and the Klein operator (or kleinian)  $\hat{k}$  obeying  $\hat{k}^2 = \mathbb{1}$ . The matrix of components  $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$  is given by  $\epsilon_{12} = 1$ . The above deformed oscillators can indeed be used to realise  $\mathfrak{sl}(2, \mathbb{R}) \cong \mathfrak{so}(1, 2) \cong \mathfrak{sp}(2, \mathbb{R})$  by defining the generators

$$T_{\alpha\beta} = \frac{1}{4i} \{\hat{q}_\alpha, \hat{q}_\beta\} \quad (1.5)$$

that obey

$$[T_{\alpha\beta}, T_{\gamma\delta}] = 2 (\epsilon_{\alpha(\gamma} T_{\delta)\beta} + \epsilon_{\beta(\gamma} T_{\delta)\alpha}) , \quad (1.6)$$

where indices inside brackets are symmetrised with strength one. The deformed oscillators can also be used to present  $\mathfrak{osp}(2|2)$  upon defining

$$Q_\alpha^{(1)} := \hat{q}_\alpha, \quad Q_\alpha^{(2)} := \hat{k} \hat{q}_\alpha, \quad J := \frac{1}{2}(\hat{k} + \nu), \quad (1.7)$$

yielding [28]

$$\begin{aligned} [T_{\alpha\beta}, Q_\gamma^{(i)}] &= -2\epsilon_{\gamma(\alpha} Q_{\beta)}^{(i)}, & \{Q_\alpha^{(i)}, Q_\beta^{(j)}\} &= 4i \left( \sigma_3^{ij} T_{\alpha\beta} - \tau^{ij} \epsilon_{\alpha\beta} J \right), \\ [T_{\alpha\beta}, J] &= 0, & [J, Q_\alpha^{(i)}] &= \tau^{ij} Q_\alpha^{(j)}, \end{aligned} \quad (1.8)$$

with  $\tau^{ij} = -\tau^{ji}$ ,  $\tau^{12} = 1$ ; and  $\sigma_3$  is the third Pauli matrix. Both restricted set of generators  $\{T_{\alpha\beta}, Q_\alpha^{(i)}\}$ , with  $i$  equals to 1 or 2, span an  $\mathfrak{osp}(1|2)$  subalgebra in  $\mathfrak{osp}(2|2)$ . An advantage using deformed oscillators is that they automatically enforce the quotient in (1.1), as well as:

$$\mathbb{C} \oplus \mathfrak{hs}[\lambda] = \frac{\mathcal{U}(\mathfrak{osp}(1|2))}{\langle \mathcal{C}_2 - \frac{1}{4}\lambda(\lambda-1)\mathbb{1} \rangle}, \quad (1.9)$$

defining the fermionic extension of  $\mathfrak{hs}[\lambda]$ , containing generators of all (half-)integer spins  $s \geq 3/2$  and where  $\mathcal{C}_2$  here denotes the  $\mathfrak{osp}(1|2)$  quadratic Casimir.

The higher spin algebra is  $\mathfrak{hs}[\lambda]$  is obtained by considering the commutators in the associative algebra made out of all possible *even* powers of the deformed oscillators, endowed with the associative *deformed*<sup>5</sup> star product:

$$q_\alpha \star q_\beta = q_\alpha q_\beta + i\epsilon_{\alpha\beta}(1 + k\nu), \quad (1.10)$$

where now  $q_\alpha$  and  $k$  (without hats) are the so-called *symbols* associated with the operators  $\hat{q}_\alpha$  and  $\hat{k}$ . Very schematically, the main idea of the symbol calculus goes as follows. To any operator  $\hat{A}(\hat{q}, \hat{k})$  one associates a classical function  $A_{\mathcal{O}}(q_\alpha, k) \equiv [\hat{A}(\hat{q}, \hat{k})]_{\mathcal{O}}$  obtained by (i) first ordering the operators  $\hat{q}_\alpha$  and  $\hat{k}$  entering the expression of  $\hat{A}(\hat{q}, \hat{k})$  by following a given ordering prescription  $\mathcal{O}$  and making

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<sup>5</sup>The terminology “deformed star product” refers to both the fact that this product is defined on the deformed oscillators, but also to the fact that, upon setting the parameter  $\nu$  to 0, one recovers the usual Moyal star product.

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use of the relations (1.4), and (ii) by then replacing the operators  $\hat{q}_\alpha$  and  $\hat{k}$  by the classical, i.e. commuting *symbols*  $q_\alpha$  and  $k$ . The composition of operators ordered according to the prescription  $\mathcal{O}$  is represented, in the (appropriately defined) space of classical symbols, by an associative but noncommutative star-product  $\star_{\mathcal{O}}$  such that

$$[\hat{A}\hat{B}]_{\mathcal{O}} = A_{\mathcal{O}} \star_{\mathcal{O}} B_{\mathcal{O}}. \quad (1.11)$$

For more precise statements and references, see e.g. [29] and [30] in the context of higher-spin theories. The ordering prescription we will be considering here is the Weyl ordering, whereby all the operators  $\hat{q}_\alpha$  are symmetrised before being replaced by their classical, commuting symbols  $q_\alpha$ . The above formula (1.10) therefore defines the star product in the Weyl ordering prescription, where the right-hand side corresponds to the sum of the anticommutator with the commutator of the operators  $\hat{q}_\alpha$  and  $\hat{q}_\beta$ . As it is customary in the present context, before replacing the operators by their symbols, one chooses to place the kleinian  $\hat{k}$  either to the left or to the right of any monomial in the  $\hat{q}_\alpha$ 's. In this paper, we will always place  $\hat{k}$  to the left.

The set of arbitrary Weyl-ordered monomials in the deformed oscillators  $q_\alpha$  together with the identity (the monomial of degree zero) and  $k$  gives a basis of the universal enveloping algebra of  $\mathfrak{osp}(1|2)$  [4]. This is an associative algebra, by construction. If one endows it with the star-commutator, one gets a Lie (super)algebra denoted  $\mathfrak{shs}[\lambda]$  [3] where the  $\mathbb{Z}_2$  grading is given by the order of a monomial in the  $q_\alpha$ , modulo 2 [4]. The structure constants of the algebra  $\mathfrak{hs}[\lambda]$  were conjectured to be given by the commutator lone-star product of [31, 32, 33, 34], see more recently the appendix B of [35] where some evidences for this conjecture were given up to spin 4.

The present paper aims at explicitly computing the structure constants appearing in the star product of two arbitrary (even or odd) Weyl-ordered monomials in the deformed oscillators. In particular, we will *prove* that the lone-star product produces the correct structure constants for the bosonic sector of the associative algebra  $Aq(2;\nu)$  underlying  $\mathfrak{hs}[\lambda]$ , and this without making any restriction on the value of the spin of the generators involved. What we understood of the interesting recent work [36] is that the lone-star product was assumed to be the underlying product for  $\mathfrak{hs}[\lambda]$  and was extended to  $Aq(2;\nu)$  by using associativity, which the author then proved in the appendix in [36]. In other words, by conjecturing the lone-star product for the algebra  $Aq(2;\nu)$ , the author proved the associativity property. In the present paper, we proceed differently by starting from the associative algebra  $Aq(2;\nu)$  and showing that the lone-star product formula is the *unique* solution for the structure constants.

Our paper is organised as follows: In Section 2, we derive a few lemmas on the star product of deformed oscillators and use them to obtain a closed formula for the searched-for structure constants. The Section 3 consists of a brief review of  $\mathcal{W}$  algebras and their connections to (three dimensional)

higher spin theories. This connection is exploited in [Section 4](#) to derive the structure constants of  $\mathfrak{hs}[\lambda]$ . In the bosonic restriction, the structure constants can be rewritten in terms of (generalised) hypergeometric functions, as postulated by Pope, Romans and Shen [\[31\]](#) in a different context.

## 2 Some deformed oscillator algebra

After recalling some basic facts about the deformed oscillators algebra, we proceed in the following section to prove some lemmas that will be needed in order to derive the structure constant of  $\mathfrak{hs}[\lambda]$ .

### 2.1 Simple (anti)commutators.

Taking as a starting point the deformed oscillator star-commutation relation:

$$[q_\alpha, q_\beta]_\star = 2i \epsilon_{\alpha\beta} (1 + \nu k) \quad , \quad \{k, q_\alpha\}_\star = 0 \quad , \quad (2.12)$$

we would like to compute the star product of  $q_\alpha$  with the completely symmetrised (*i.e.* Weyl-ordered) product

$$q_{\beta_1} \dots q_{\beta_n} := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} q_{\beta_{\sigma(1)}} \star q_{\beta_{\sigma(2)}} \star \dots \star q_{\beta_{\sigma(n)}} \quad , \quad (2.13)$$

where  $\mathfrak{S}_n$  denotes the group of permutations of  $n$  elements. The usual decomposition rule (Pieri's rule) for the tensor product of irreducibles of  $\mathfrak{S}_n$  into irreducibles of  $\mathfrak{S}_n$  gives

$$\begin{aligned} q_\alpha \star (q_{\beta_1} \dots q_{\beta_n}) &= q_\alpha q_{\beta_1} \dots q_{\beta_n} + \frac{2}{(n+1)} \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \left[ \overset{\frown}{q_\alpha} \star \overset{\smile}{q_{\beta_{\sigma(1)}}} \star q_{\beta_{\sigma(2)}} \dots \star q_{\beta_{\sigma(n)}} + \right. \\ &\quad \left. + \overset{\frown}{q_\alpha} \star q_{\beta_{\sigma(2)}} \star \overset{\smile}{q_{\beta_{\sigma(1)}}} \star \dots \star q_{\beta_{\sigma(n)}} + \dots + \overset{\frown}{q_\alpha} \star q_{\beta_{\sigma(2)}} \star \dots \star q_{\beta_{\sigma(n)}} \star \overset{\smile}{q_{\beta_{\sigma(1)}}} \right] \end{aligned} \quad (2.14)$$

where we introduced the following notation for antisymmetrization:

$$\overset{\frown}{a_1} \overset{\smile}{a_2} \dots \overset{\frown}{a_i} \overset{\smile}{a_{i+1}} \dots a_n = \frac{1}{2} \left( a_1 a_2 \dots a_{i-1} a_i a_{i+1} \dots a_n - a_i a_2 \dots a_{i-1} a_1 a_{i+1} \dots a_n \right) \quad . \quad (2.15)$$

In Eq. (2.14), it is possible to bring next to each other every two elements  $q_\alpha$  and  $q_{\beta_{\sigma(1)}}$  that are antisymmetrized in the sum, so as to produce a commutator  $[q_\alpha, q_{\beta_{\sigma(1)}}]_\star$ . This can be done by dragging a term sitting at the place  $i$  in the chain of star product to the place 2, thereby producing extra terms with commutators. Summing up everything together, one finds

$$\begin{aligned} q_\alpha \star (q_{\beta_1} \dots q_{\beta_n}) &= q_\alpha q_{\beta_1} \dots q_{\beta_n} + \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \frac{2i}{(n+1)} \times \\ &\quad \times \left[ \sum_{i=0}^{n-1} (n-i) q_{\beta_{\sigma(1)}} \star \dots \star q_{\beta_{\sigma(i)}} \star (1 + \nu k) \epsilon_{\alpha\beta_{\sigma(i+1)}} \star q_{\beta_{\sigma(i+2)}} \star \dots \star q_{\beta_{\sigma(n)}} \right] . \end{aligned} \quad (2.16)$$

At this stage, it is a matter of performing the sums corresponding to the “1” and to the “ $\nu k$ ” in the  $(1 + \nu k)$ 's appearing at the different places in the chain of star product. The first sum is easy to do:  $\sum_{i=0}^{n-1} (n-i) = \frac{n(n+1)}{2}$ . One has to be more careful with the second one (involving the kleinian  $k$ )

since it produces an alternating sum and a distinction must be done between the cases where  $n$  is even and odd. It is nevertheless straightforward and gives the final answer

$$q_\alpha \star (q_{\beta_1} \dots q_{\beta_n}) = q_\alpha q_{\beta_1} \dots q_{\beta_n} + i n \left( 1 + \frac{2n+1-(-1)^n}{2n(n+1)} \nu k \right) \epsilon_{\alpha(\beta_1 q_{\beta_2} \dots q_{\beta_n})} \quad (2.17)$$

that correctly reproduces the case  $n = 1$ . Similarly one gets

$$(q_{\beta_1} \dots q_{\beta_n}) \star q_\alpha = q_\alpha q_{\beta_1} \dots q_{\beta_n} - i n \left( 1 + (-1)^{n+1} \frac{2n+1-(-1)^n}{2n(n+1)} \nu k \right) \epsilon_{\alpha(\beta_1 q_{\beta_2} \dots q_{\beta_n})} \quad (2.18)$$

Using these two results, one finds the following commutators and anticommutators:

$$[q_\alpha, (q_\beta)^n]_\star = 2i\epsilon_{\alpha\beta} (n + k\nu P_n) (q_\beta)^{n-1} \quad , \quad P_n := \frac{1-(-1)^n}{2} \quad (2.19)$$

$$\{q_\alpha, (q_\beta)^n\}_\star = 2q_\alpha (q_\beta)^n + 2i\epsilon_{\alpha\beta} \frac{n}{(n+1)} k\nu P_{n+1} (q_\beta)^{n-1} \quad . \quad (2.20)$$

As a corollary, we find

$$[q_\alpha, (q_\alpha)^r (q_\beta)^s]_\star = 2i\epsilon_{\alpha\beta} \frac{s}{(r+s)} [(r+s) + k\nu P_{r+s}] (q_\alpha)^r (q_\beta)^{s-1} \quad , \quad (2.21)$$

$$\{q_\alpha, (q_\alpha)^r (q_\beta)^s\}_\star = 2(q_\alpha)^{r+1} (q_\beta)^s + (2i\epsilon_{\alpha\beta}) k\nu \frac{s}{(r+s)} \frac{(r+s)}{(r+s+1)} P_{r+s+1} (q_\alpha)^r (q_\beta)^{s-1} \quad . \quad (2.22)$$

We use notation whereby repeated indices are completely symmetrized with strength one, and  $(q_\beta)^n$  stands for  $q_{\beta_1} \dots q_{\beta_n}$ . Grouping together the commutator (2.19) and anticommutator (2.20), one derives the following formula which is central in the forthcoming computations involving deformed oscillators:

$$q_\alpha \star (q_\beta)^n = q_\alpha (q_\beta)^n + i\epsilon_{\alpha\beta} n Y_n^+ (q_\beta)^{n-1} \quad , \quad Y_n^\pm := 1 \pm k\nu \left[ \frac{P_n}{n} + \frac{P_{n+1}}{n+1} \right] \quad , \quad (2.23)$$

$$q_\alpha \star [(q_\alpha)^r (q_\beta)^s] = (q_\alpha)^{r+1} (q_\beta)^s + i\epsilon_{\alpha\beta} s Y_{r+s}^+ (q_\alpha)^r (q_\beta)^{s-1} \quad . \quad (2.24)$$

A similar relation can be derived for the star product of one oscillator with a monomial from the right:

$$(q_\alpha)^n (q_\beta)^m \star q_\beta = (q_\alpha)^n (q_\beta)^{m+1} + i\epsilon_{\alpha\beta} n \bar{Y}_{n+m} (q_\alpha)^{n-1} (q_\beta)^m \quad , \quad \bar{Y}_n = 1 + k\nu \left[ \frac{P_n}{n} - \frac{P_{n+1}}{n+1} \right] \quad . \quad (2.25)$$

## 2.2 Recurrence relation

By nesting formulas (2.23) and (2.24), it is now straightforward to compute the star-product of two monomials of arbitrary degrees  $m$  and  $n$ , where without loss of generality one chooses  $m \leq n$ :

$$\begin{aligned} (q_\alpha)^m \star (q_\beta)^n &= (q_\alpha)^{m-1} \star [q_\alpha (q_\beta)^n + (i\epsilon_{\alpha\beta}) n Y_n^+ (q_\beta)^{n-1}] \\ &= (q_\alpha)^{m-2} \star \left( (q_\alpha)^2 (q_\beta)^n + (i\epsilon_{\alpha\beta}) \frac{n}{(n+1)} (n+1) Y_{n+1}^+ q_\alpha (q_\beta)^{n-1} + \right. \\ &\quad \left. (i\epsilon_{\alpha\beta}) n Y_n^- [q_\alpha (q_\beta)^{n-1} + (i\epsilon_{\alpha\beta}) (n-1) Y_{n-1}^+ (q_\beta)^{n-2}] \right) = \dots \end{aligned} \quad (2.26)$$

where one continues  $(m-2)$  times until one has exhausted the degree of the first monomial. Writing the above star product as<sup>6</sup>:

$$(q_\alpha)^m \star (q_\beta)^n = \sum_{p=0}^m \frac{(i\epsilon_{\alpha\beta})^p n!}{(n-p)!} b_p^{(m,n)} (q_\alpha)^{m-p} (q_\beta)^{n-p} \quad , \quad (2.27)$$

our problem then boils down to computing the structure constants  $b_p^{(m,n)}$ .

<sup>6</sup>Notice that in the general case, the upper bound of the sum in (2.27) is  $\min(m, n)$ .

**From the left.** To do so, we will start by using the associativity of the star product to extract a recurrence relation on the coefficients  $b_p^{(m,n)}$ . Defining  $\pi(\sum_k \prod_i Y_{n_{k,i}}^{\epsilon_{k,i}}) = \sum_k \prod_i Y_{n_{k,i}}^{-\epsilon_{k,i}}$  (all signs of the  $Y_n^\pm$  symbols are flipped, i.e.  $k\nu \rightarrow -k\nu$ ), we have:

$$\begin{aligned}
(q_\alpha)^{m+1} \star (q_\beta)^n &= q_\alpha \star (q_\alpha)^m \star (q_\beta)^n = q_\alpha \star \sum_{p=0}^m \frac{(i\epsilon_{\alpha\beta})^p n!}{(n-p)!} b_p^{(m,n)} (q_\alpha)^{m-p} (q_\beta)^{n-p} \\
&= \sum_{p=0}^m \frac{(i\epsilon_{\alpha\beta})^p n!}{(n-p)!} \pi(b_p^{(m,n)}) \left( (q_\alpha)^{m-p+1} (q_\beta)^{n-p} + (i\epsilon_{\alpha\beta})(n-p) Y_{n+m-2p}^+ (q_\alpha)^{m-p} (q_\beta)^{n-p-1} \right) \\
&= \sum_{p=0}^m \frac{(i\epsilon_{\alpha\beta})^p n!}{(n-p)!} \pi(b_p^{(m,n)}) (q_\alpha)^{m-p+1} (q_\beta)^{n-p} \\
&\quad + \sum_{p=1}^{m+1} \frac{(i\epsilon_{\alpha\beta})^p n!}{(n-p)!} \pi(b_{p-1}^{(m,n)}) Y_{n+m-2(p-1)}^+ (q_\alpha)^{m+1-p} (q_\beta)^{n-p} \\
&= \sum_{p=0}^{m+1} \frac{(i\epsilon_{\alpha\beta})^p n!}{(n-p)!} b_p^{(m+1,n)} (q_\alpha)^{m+1-p} (q_\beta)^{n-p}, \tag{2.28}
\end{aligned}$$

$$\Rightarrow \begin{cases} b_0^{(m,n)} = 1, \forall m, n \in \mathbb{N}, & b_m^{(m,n)} = \prod_{i=0}^{m-1} Y_{n-i}^{(-1)^{m+1+i}}, \forall m, n \in \mathbb{N} \\ b_p^{(m+1,n)} = \pi(b_p^{(m,n)}) + Y_{n+m-2(p-1)}^+ \pi(b_{p-1}^{(m,n)}), \forall m, n \in \mathbb{N}, \forall p \in \llbracket 1, m \rrbracket \end{cases}. \tag{2.29}$$

Notice that, in the special case  $\nu = 0$ , this recurrence relation reduces to the one obeyed by the binomial coefficients:

$$\binom{m+1}{p} = \binom{m}{p} + \binom{m}{p-1}. \tag{2.30}$$

Indeed, for  $\nu = 0$ ,  $\pi(b_p^{(m,n)}) = b_p^{(m,n)}$  and  $Y_n^\pm = 1$ . This was to be expected, as one should recover the Moyal star product, for which the coefficients of  $(i\epsilon_{\alpha\beta})^p$  in (2.27) is  $\frac{1}{p!} \frac{n!}{(n-p)!} \frac{m!}{(m-p)!}$ . The factor  $\frac{1}{p!}$  comes from the Taylor coefficient of the exponential in the Moyal star-product formula, while the other two contributions  $\frac{m!}{(m-p)!}$  and  $\frac{n!}{(n-p)!}$  come from taking  $p$  derivatives of the monomials of order  $m$  and  $n$ , respectively.

After some brute force computation, one can guess from several examples the following general formula:

$$b_p^{(m,n)} = \prod_{\ell=0}^{p-1} \sum_{\substack{i_\ell=i_{\ell-1} \\ i_0=0}}^{m-p} Y_{n+i_\ell-\ell}^{(-1)^{m+i_\ell+\ell+1}}, \tag{2.31}$$

or in a less compact way:

$$b_p^{(m,n)} = \sum_{i_0=0}^{m-p} \sum_{i_1=i_0}^{m-p} \cdots \sum_{i_{p-1}=i_{p-2}}^{m-p} Y_{n+i_0}^{(-1)^{m+i_0+1}} Y_{n+i_1-1}^{(-1)^{m+i_1+2}} \cdots Y_{n+i_{p-1}-(p-1)}^{(-1)^{m+i_{p-1}+p}}. \tag{2.32}$$

This formula reproduces the coefficients calculated above, and more importantly verifies the recurrence relation (2.29).



*Proof.* Let us first write:

$$\bullet \quad \pi(b_p^{(m,n)}) = \sum_{i_0=0}^{m-p} \sum_{i_1=i_0}^{m-p} \cdots \sum_{i_{p-1}=i_{p-2}}^{m-p} Y_{n+i_0}^{(-1)^{m+i_0+2}} Y_{n+i_1-1}^{(-1)^{m+i_1+3}} \cdots Y_{n+i_{p-1}-(p-1)}^{(-1)^{m+i_{p-1}+p+1}} ; \quad (2.33)$$

$$\bullet \quad Y_{n+m-2(p-1)}^+ \pi(b_{p-1}^{(m,n)}) = Y_{n+m-2(p-1)}^+ \sum_{i_0=0}^{m-p+1} \sum_{i_1=i_0}^{m-p+1} \cdots \sum_{i_{p-2}=i_{p-3}}^{m-p+1} Y_{n+i_0}^{(-1)^{m+i_0+2}} \cdots Y_{n+i_{p-2}-(p-2)}^{(-1)^{m+i_{p-2}+p}} . \quad (2.34)$$

Noticing that  $Y_{n+m-2(p-1)}^+ = Y_{n+i_{p-1}-(p-1)}^{(-1)^{m+i_{p-1}+p+1}} \delta_{i_{p-1}, m-p+1}$ , and denoting by the symbol  $\Pi_{p-1}$  the product  $Y_{n+i_0}^{(-1)^{m+i_0+2}} Y_{n+i_1-1}^{(-1)^{m+i_1+3}} \cdots Y_{n+i_{p-1}-(p-1)}^{(-1)^{m+i_{p-1}+p+1}}$ , the sum of the two terms above reads:

$$\pi(b_p^{(m,n)}) + Y_{n+m-2(p-1)}^+ \pi(b_{p-1}^{(m,n)}) = \left( \sum_{i_0=0}^{m-p} \cdots \sum_{i_{p-1}=i_{p-2}}^{m-p} + \sum_{i_0=0}^{m-p+1} \cdots \sum_{i_{p-2}=i_{p-3}}^{m-p+1} \delta_{i_{p-1}, m-p+1} \right) \Pi_{p-1} . \quad (2.35)$$

On the other hand, we have:

$$\begin{aligned} b_p^{(m+1,n)} &= \sum_{i_0=0}^{m-p+1} \sum_{i_1=i_0}^{m-p+1} \cdots \sum_{i_{p-1}=i_{p-2}}^{m-p+1} Y_{n+i_0}^{(-1)^{m+i_0+2}} Y_{n+i_1-1}^{(-1)^{m+i_1+3}} \cdots Y_{n+i_{p-1}-(p-1)}^{(-1)^{m+i_{p-1}+p+1}} \\ &= \left( \sum_{i_0=0}^{m-p+1} \sum_{i_1=i_0}^{m-p+1} \cdots \sum_{i_{p-1}=i_{p-2}}^{m-p+1} \right) \Pi_{p-1} \\ &= \left( \sum_{i_0=0}^{m-p} \sum_{i_1=i_0}^{m-p+1} \cdots \sum_{i_{p-1}=i_{p-2}}^{m-p+1} + \delta_{i_0, m-p+1} \cdots \delta_{i_{p-1}, m-p+1} \right) \Pi_{p-1} \\ &= \left( \sum_{i_0=0}^{m-p} \sum_{i_1=i_0}^{m-p} \sum_{i_2=i_1}^{m-p+1} \cdots \sum_{i_{p-1}=i_{p-2}}^{m-p+1} + \right. \\ &\quad \left. + \sum_{i_0=0}^{m-p} \delta_{i_1, m-p+1} \cdots \delta_{i_{p-1}, m-p+1} + \delta_{i_0, m-p+1} \cdots \delta_{i_{p-1}, m-p+1} \right) \Pi_{p-1} \\ &\quad \quad \quad = \sum_{i_0=0}^{m-p+1} \delta_{i_1, m-p+1} \cdots \delta_{i_{p-1}, m-p+1} \\ &= \cdots = \left( \sum_{i_0=0}^{m-p} \sum_{i_1=i_0}^{m-p} \cdots \sum_{i_{p-1}=i_{p-2}}^{m-p} + \sum_{i_0=0}^{m-p+1} \cdots \sum_{i_{p-2}=i_{p-3}}^{m-p+1} \delta_{i_{p-1}, m-p+1} \right) \Pi_{p-1} \\ &= \pi(b_p^{(m,n)}) + Y_{n+m-2(p-1)}^+ \pi(b_{p-1}^{(m,n)}) , \end{aligned} \quad (2.36)$$

i.e. the coefficients given by (2.31) verify the recurrence relation (2.29).  $\square$

**Reduced formula.** Although formula (2.31) is exact, it is almost impossible to use in actual computations. However, for particular values of the indice  $p$ , it can be reduced and written in a more compact form.

- For  $p = 1$ , all the products in (2.31) collapse and one is left with just an alternating sum:

$$b_1^{(m,n)} = \sum_{\ell=0}^{m-1} Y_{n+\ell}^{(-1)^{m+1+\ell}} \quad (2.37)$$

$$= \sum_{\ell=0}^{m-1} 1 + (-1)^{m+1+\ell} k\nu \left( \frac{P_{n+\ell}}{n+\ell} + \frac{P_{n+\ell+1}}{n+\ell+1} \right) \quad (2.38)$$

$$= m + k\nu \left( (-1)^{m+1} \frac{P_n}{n} + \frac{P_{n+m}}{n+m} \right) ; \quad (2.39)$$

- For  $p = m$ , all the sums in (2.31) collapse and one is left with:

$$b_m^{(m,n)} = \prod_{\ell=0}^{m-1} Y_{n-\ell}^{(-1)^{m+1+\ell}} \quad (2.40)$$

$$= \prod_{\ell=0}^{m-1} \left( 1 - (-1)^{m+\ell} \frac{k\nu(n-\ell+P_{n-\ell})}{(n-\ell)(n-\ell+1)} \right) \quad (2.41)$$

$$= \prod_{\ell=0}^{m-1} \left( \frac{(n-\ell+P_{n-\ell})(n-\ell+1-P_{n-\ell}-(-1)^{m-\ell}k\nu)}{(n-\ell)(n-\ell+1)} \right). \quad (2.42)$$

One can then show, by considering separate cases for the parity of  $m$  and  $n$ , the following identities:

$$\prod_{\ell=0}^{m-1} (n-\ell+1-P_{n-\ell}-(-1)^{m-\ell}k\nu) = 2^m \left[ \frac{n-k\nu+(-1)^m P_{n+1}}{2} \right]_{\frac{m-P_m}{2}} \left[ \frac{n+k\nu-(-1)^m P_{n+1}}{2} \right]_{\frac{m+P_m}{2}} \quad (2.43)$$

and

$$\prod_{\ell=0}^{m-1} \frac{(n-\ell+P_{n-\ell})}{(n-\ell)(n-\ell+1)} = \frac{2^{-m}}{\left[ \frac{n+P_{n+1}}{2} \right]_{\frac{m+P_m}{2}} \left[ \frac{n-P_{n+1}}{2} \right]_{\frac{m-P_m}{2}}}, \quad (2.44)$$

which give, altogether:

$$b_m^{(m,n)} = \frac{\left[ \frac{n-k\nu+(-1)^m P_{n+1}}{2} \right]_{\frac{m-P_m}{2}} \left[ \frac{n+k\nu-(-1)^m P_{n+1}}{2} \right]_{\frac{m+P_m}{2}}}{\left[ \frac{n+P_{n+1}}{2} \right]_{\frac{m+P_m}{2}} \left[ \frac{n-P_{n+1}}{2} \right]_{\frac{m-P_m}{2}}}, \quad (2.45)$$

where we used the *descending Pochhammer symbol* defined as:  $[a]_n = \frac{\Gamma(a+1)}{\Gamma(a-n+1)}$ ,  $a \in \mathbb{R}$ ,  $n \in \mathbb{N}$ .

**From the right.** With again  $n \geq m$ , we can also look at what happens when the monomial of highest degree is on the left — and hence product have to be performed toward the left<sup>7</sup>. As we did previously, we start by writing the expansion of the star product of two arbitrary monomials as:

$$(q_\alpha)^n \star (q_\beta)^m = \sum_{p=0}^m \frac{(i\epsilon_{\alpha\beta})^p n!}{(n-p)!} \bar{b}_p^{(n,m)} (q_\alpha)^{n-p} (q_\beta)^{m-p}. \quad (2.46)$$

<sup>7</sup>This difference between the structure constants produced when the highest degree monomial is placed on the left or on the right stems from the fact that, as explained in the Introduction, we choose to place the Kleinian on the left of the resulting monomials.

Once again, a recurrence relation can be derived for these  $\bar{b}_p^{(n,m)}(\nu)$  coefficients:

$$\begin{aligned}
(q_\alpha)^n \star (q_\beta)^{m+1} &= \left( \sum_{p=0}^m \frac{(i\epsilon_{\alpha\beta})^p n!}{(n-p)!} \bar{b}_p^{(n,m)} (q_\alpha)^{n-p} (q_\beta)^{m-p} \right) \star q_\beta \\
&= \sum_{p=0}^m \frac{(i\epsilon_{\alpha\beta})^p n!}{(n-p)!} \bar{b}_p^{(n,m)} [(q_\alpha)^{n-p} (q_\beta)^{m-p+1} + i\epsilon_{\alpha\beta} (n-p) \bar{Y}_{n+m-2p} (q_\alpha)^{n-1-p} (q_\beta)^{m-p}] \\
&= \sum_{p=0}^m \frac{(i\epsilon_{\alpha\beta})^p n!}{(n-p)!} \bar{b}_p^{(n,m)} (q_\alpha)^{n-p} (q_\beta)^{m-p+1} \\
&\quad + \sum_{p=1}^{m+1} \frac{(i\epsilon_{\alpha\beta})^p n!}{(n-p)!} \bar{b}_{p-1}^{(n,m)} \bar{Y}_{n+m-2(p-1)} (q_\alpha)^{n-p} (q_\beta)^{m-p+1} \\
&= \sum_{p=0}^{m+1} \frac{(i\epsilon_{\alpha\beta})^p n!}{(n-p)!} \bar{b}_p^{(n,m+1)} (q_\alpha)^{n-p} (q_\beta)^{m-p+1}, \tag{2.47}
\end{aligned}$$

$$\Rightarrow \bar{b}_p^{(n,m+1)} = \bar{b}_p^{(n,m)} + \bar{Y}_{n+m-2(p-1)} \bar{b}_{p-1}^{(n,m)}, \quad \forall p \leq m \leq n \in \mathbb{N}. \tag{2.48}$$

Exactly as in the previous case, an exact expression for these coefficients is given by sums of products of  $\bar{Y}$ 's:

$$\bar{b}_p^{(n,m)} = \prod_{\ell=0}^{p-1} \sum_{\substack{i_\ell=i_{\ell-1} \\ i_0=0}}^{m-p} \bar{Y}_{n+i_\ell-\ell}. \tag{2.49}$$

It can be checked that this expression solves the recurrence relation given above, the proof being essentially the same as for the coefficients  $b_p^{(m,n)}$ .

**Reduced formula.** Due to its structure, similar to (2.31), (2.49) can be simplified in two limit cases:

- For  $p = 1$ , one is only left with a sum that can be performed explicitly:

$$\bar{b}_1^{(n,m)} = m + k\nu \left( \frac{P_n}{n} - \frac{P_{n+m}}{n+m} \right); \tag{2.50}$$

- For  $p = m$ , only the product remains, which can also be computed, considering separate cases according to the respective parity of  $m$  and  $n$ :

$$\bar{b}_m^{(n,m)} = \frac{\left[ \frac{n+k\nu-P_{n+1}}{2} \right]_{\frac{m-(-1)^n P_m}{2}} \left[ \frac{n-k\nu+P_{n+1}}{2} \right]_{\frac{m+(-1)^n P_m}{2}}}{\left[ \frac{n-P_{n+1}}{2} \right]_{\frac{m-P_m}{2}} \left[ \frac{n+P_{n+1}}{2} \right]_{\frac{m+P_m}{2}}}. \tag{2.51}$$

### 3 $\mathcal{W}$ algebras interlude

Before relating the structure constants  $b_p^{(2m,2n)}$  (of  $\mathfrak{hs}[\lambda]$ ) to those postulated in [31], we succinctly recall throughout the following section what are  $\mathcal{W}$  algebras and their relation to 3D higher spin algebras. For a more complete introduction to  $\mathcal{W}$  algebras, see e.g. [37].

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$\mathcal{W}_{N,c}$  algebras naturally appear in the context of 2D conformal field theories involving higher spin currents [38], i.e. with spins  $s \in \llbracket 2, N \rrbracket$ , and central charge  $c$ . They can be thought of as higher spin extensions of the Virasoro algebra (describing a spin 2 quasi-primary current, namely the stress-energy tensor) in this context. Because of the non linear terms appearing in the Operator Product Expansion (OPE) of such higher spin currents, the structure of the corresponding  $\mathcal{W}_{N,c}$  algebras becomes quite intricate and, in particular, they are not Lie algebras.

More recently,  $\mathcal{W}$  algebras appeared as algebras of asymptotic symmetries of 3D higher spin theories [12, 39, 40, 41]. When the higher spin theory involves an infinite tower of gauge fields with all spin  $s \in \mathbb{N}$ , as in Prokushkin–Vasiliev’s theory or the Chern–Simons theory based on  $\mathfrak{hs}[\lambda] \oplus \mathfrak{hs}[\lambda]$ , the asymptotic symmetry algebra is an infinite dimensional extension of the  $\mathcal{W}_{N,c}$  algebras, referred to as  $\mathcal{W}_{\infty,c}$ , which corresponds to the algebra made out of all higher spin currents together with the stress-energy tensor. A first attempt to obtain such an extension was carried out in [42]<sup>8</sup> where the authors obtained a Lie (hence linear) algebra, that we shall refer to as  $\mathcal{W}_{\infty}^{\text{PRS}}$  hereafter, further explored in [32, 33]. In a later paper [31], the authors realised that there was a one-parameter family of such algebras, which their first construction was a part of. They showed that for each value of  $\mu$  (a real number parametrising their family of extension), these algebras admit a subalgebra, called *wedge subalgebra*. They found out that this subalgebra is isomorphic to  $\mathfrak{hs}[\lambda]$ , where  $\mu = \frac{\lambda^2 - 1}{4}$  is the value of the quadratic Casimir of  $\mathfrak{sl}(2, \mathbb{R})$ , as both can be seen as the quotient (1.1). A puzzling feature of their construction for these *linear* infinite dimensional extensions of  $\mathcal{W}_N$  algebras is the fact that the introduction of an infinite number of generators carrying negative spin is needed in order to satisfy the Jacobi identity, except for the special value  $\mu = 0$ . For this value, the resulting, *linear* infinite-dimensional algebra is the Lie algebra denoted  $\mathcal{W}_{\infty}^{\text{PRS}}$ . For  $\mu \neq 0$ , it is still not clear to us whether the resulting algebras containing negative-spin generators can be related to *nonlinear*  $\mathcal{W}_{\infty,c}$ .

A more modern point of view in obtaining such extensions is given by, from a mathematical point of view, the *Drinfeld–Sokolov reduction*, which associates to a semisimple Lie algebra a centrally extended  $\mathcal{W}$  algebra. This operation corresponds, from a physical point of view [44], to the passage of the gauge algebra of some theory defined around anti-de Sitter (AdS) background to its asymptotic symmetry algebra; see for instance [45, 41] for enlightening reviews of the interplay between the two approaches. The  $\mathcal{W}_{\infty}$  algebras obtained via the asymptotic symmetry algebra procedure do not suffer from the odd feature of having negative spin generators, and, for generic values of the parameter  $\lambda$  of the higher-spin gauge algebra  $\mathfrak{hs}[\lambda]$ , are nonlinear — except for  $\lambda = \pm 1$ , i.e.  $\mu = 0$  [46].

In spite of the difficulties brought in by the appearance of nonlinear terms, the structure constants of  $\mathcal{W}_{\infty,c}$  have been derived [41] in terms of those of  $\mathfrak{hs}[\lambda]$ , which is expected to coincide with its wedge

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<sup>8</sup>Actually the first appearance of such an higher spin extension was given in [43], which was later realised to be a particular contraction of the Pope, Shen and Romans  $\mathcal{W}_{\infty}^{\text{PRS}}$ , denoted latter on as  $w_{\infty}$ .

in the limit  $c \rightarrow \infty$ . The structure constants of  $\mathfrak{hs}[\lambda]$  were postulated in [31], in the Fourier basis given by the generators  $V_m^s$  carrying spin  $s + 2$ , with  $|m| \leq s + 1$ . These generators verify:

$$[V_m^i, V_n^j] = \sum_{\ell=0}^{\infty} g_{2\ell}^{i,j}(m, n) V_{m+n}^{i+j-2\ell}, \quad (3.52)$$

where  $g_{2\ell}^{i,j}(m, n)$  are the structure constants given hereafter. It was shown [31] that this commutation relations could be realised as the antisymmetric part of an associative algebra spanned by the same generators  $V_m^s$  and endowed with an associative product, the so-called “lone-star product”:

$$V_m^i \star V_n^j = \frac{1}{2} \sum_{a=0}^{\infty} q^{a-1} g_{a-1}^{i,j}(m, n; \lambda) V_{m+n}^{i+j-a+1}, \quad (3.53)$$

with

$$g_a^{i,j}(m, n; \lambda) = \frac{1}{2(a+1)!} N_a^{i,j}(m, n) \phi_a^{i,j}(\lambda), \quad (3.54)$$

$$N_a^{i,j}(m, n) = \sum_{r=0}^{a+1} (-1)^r \binom{a+1}{r} [i+1+m]_{a+1-r} [i+1-m]_r [j+1+n]_r [j+1-n]_{a+1-r}, \quad (3.55)$$

and

$$\phi_a^{i,j}(\lambda) = {}_4F_3 \left[ \begin{matrix} \frac{1-2\lambda}{2}, \frac{1+2\lambda}{2}, -\frac{a+1}{2}, -\frac{a}{2} \\ -\frac{2i+1}{2}, -\frac{2j+1}{2}, i+j-a+\frac{5}{2} \end{matrix}; 1 \right]. \quad (3.56)$$

The parameter  $q$  can be used to rescale<sup>9</sup> the generators  $V_m^s$ . The  $\mathfrak{sl}(2, \mathbb{R})$  algebra is generated by the  $V_m^s$  with  $s = 0$ , and its generators denoted by  $\{J_-, J_0, J_+\}$ , obey

$$[J_+, J_-] = 2J_0, \quad [J_{\pm}, J_0] = \pm J_{\pm}, \quad (3.57)$$

with  $J_0^\dagger = J_0$  and  $J_{\pm}^\dagger = J_{\mp}$ , in accordance with  $(V_m^s)^\dagger = V_{-m}^s$ .

## 4 Deriving the structure constants of $\mathfrak{shs}[\lambda]$

In this section, we build the various powers of the  $J_0$  generator of  $\mathfrak{sl}(2, \mathbb{R})$  out of deformed oscillators and prove that the structure constants postulated in [31] are indeed those of  $\mathfrak{hs}[\lambda]$ , or equivalently the wedge subalgebra of  $\mathcal{W}_\infty$ . We extend the result to the  $\mathbb{Z}_2$ -graded case of  $\mathfrak{shs}[\lambda]$ .

### 4.1 Dictionary with the wedge subalgebra

We define  $w$  as

$$w = a^+ a^- = \frac{1}{2} \{a^-, a^+\}_\star, \quad [w, a^\pm]_\star = \pm a^\pm, \quad (4.58)$$

where  $a^+$  and  $a^-$  are the deformed creation and annihilation operators, defined as

$$a^\pm = u^{\pm\alpha} q_\alpha, \quad u^{+\alpha} u_\alpha^- = \epsilon^{\alpha\beta} u_\alpha^- u_\beta^+ = -\frac{i}{2}, \quad (u_\alpha^\pm)^\dagger = u_\alpha^\mp. \quad (4.59)$$

<sup>9</sup>It is in the limit  $q \rightarrow 0$  that one recovers the algebra studied in [43].

These operators obey

$$[a^-, a^+]_\star = 1 + \nu k, \quad \{k, a^\pm\}_\star = 0, \quad (a^\pm)^\dagger = a^\mp, \quad (4.60)$$

and can be used to realise  $\mathfrak{sl}(2, \mathbb{R})$ , by defining  $J_0 = \frac{1}{2}w$  and  $J_\pm = \frac{1}{2}(a^\mp)^2$ . The parameter  $\nu$  of the deformed oscillators can be related to  $\lambda$ , used to express the  $\mathfrak{sl}(2, \mathbb{R})$  quadratic Casimir by comparing (1.1) to the expression obtained when  $\mathfrak{sl}(2, \mathbb{R})$  is realised by these oscillators:

$$\mathcal{C}_2[\mathfrak{sl}(2, \mathbb{R})]_{\text{osc.}} = \frac{1}{16}(k\nu - 3)(k\nu + 1) = \frac{1}{16}(\nu^2 - 2k\nu - 3). \quad (4.61)$$

At this point, further comments on the relation between  $Aq(2; \nu)$  and  $\mathfrak{hs}[\lambda]$  are in order. So far, we worked entirely in  $Aq(2; \nu)$ , i.e. with arbitrary powers of the deformed oscillators  $q$  and the Kleinian  $k$ . As mentioned earlier, the associative algebra admits a subalgebra  $Aq(2; \nu)_e$  consisting of even-degree monomials in  $q$ , together with powers 0 or 1 of  $k$ . This subalgebra can be further decomposed into two consistent subalgebras, by projecting it using  $\Pi_\pm := \frac{1 \pm k}{2} : Aq(2; \nu)_e = \Pi_+ Aq(2; \nu)_e \oplus \Pi_- Aq(2; \nu)_e$  [4]. Therefore, working only in one of these two projected subalgebras, the Kleinian  $k$  can be set to  $\pm 1$ . As can be seen from (1.5),  $k$  does not enter the construction of the  $\mathfrak{sl}(2, \mathbb{R})$  UEA from deformed oscillators, therefore to realise  $\mathfrak{hs}[\lambda]$  using them, we need to use only one of the two projections of  $Aq(2; \nu)_e$ . The above relation (4.61) then becomes  $\mathcal{C}_2 = \frac{1}{16}(\nu^2 \mp 2\nu - 3)$ , which leads, upon comparing it with (1.1), to  $\lambda = \frac{1 \mp \nu}{2}$ . Hereafter, when we treat the bosonic algebra  $\mathfrak{hs}[\lambda]$  we will work in the  $\Pi_+$  projection of  $Aq(2; \nu)_e$  i.e. we set  $k\nu = \nu$ , and  $\lambda = \frac{1 - \nu}{2}$ . Otherwise, in the generic case of  $\mathfrak{shs}[\lambda]$ , we keep the Klein operator  $k$  explicitly in the various expressions.

As a corollary of the formula (2.31), we can compute the star product  $w^m \star w^n$ , where to fix the ideas we consider  $m \leq n$ . We start with

$$(q_\alpha)^{2m} \star (q_\beta)^{2n} = \sum_{p=0}^{2m} \frac{(i\epsilon_{\alpha\beta})^p (2n)!}{(2n-p)!} b_p^{(2m, 2n)} (q_\alpha)^{2m-p} (q_\beta)^{2n-p} \quad (4.62)$$

and contract both sides with

$$(u^{+\alpha})^m (u^{-\alpha})^m (u^{+\beta})^n (u^{-\beta})^n. \quad (4.63)$$

This way, the left-hand side produces  $w^m \star w^n$ . As for the right-hand side, the structures that have a chance to give a nonzero result when contracted with  $(i\epsilon_{\alpha\beta})^p (q_\alpha)^{2m-p} (q_\beta)^{2n-p}$  are

$$\sum_{r=0}^p C_{(m,n)}^{(p,r)} \left( \left[ (u^{+\alpha})^r (u^{-\beta})^r \right] \left[ (u^{+\beta})^{p-r} (u^{-\alpha})^{p-r} \right] (u^{+\alpha})^{m-r} (u^{-\alpha})^{m-p+r} (u^{-\beta})^{n-r} (u^{+\beta})^{n-p+r} \right). \quad (4.64)$$

When  $r = 0$ , one has the normalisation coefficient  $C_{(m,n)}^{(p,0)} = \frac{(2m-p)!(2n-p)!(p!)^2}{(2m)!(2n)!} \binom{m}{p} \binom{n}{p}$ . In the general case, one gets  $C_{(m,n)}^{(p,r)} = \frac{(2n-p)!}{(2n)!} \frac{(2m-p)!}{(2m)!} r!(p-r)! p! \binom{m}{r} \binom{m}{p-r} \binom{n}{r} \binom{n}{p-r}$ . Using  $[a]_n = \frac{a!}{(a-n)!} = \binom{a}{n} n!$ ,  $\forall a \in \mathbb{N}$

such that  $a \geq n$ , one can rewrite  $C_{(m,n)}^{(p,r)}$  as follows:

$$\begin{aligned}
C_{(m,n)}^{(p,r)} &= \frac{(2n-p)!}{(2n)!} \frac{(2m-p)!}{(2m)!} r!(p-r)! p! \binom{m}{r} \binom{m}{p-r} \binom{n}{r} \binom{n}{p-r} \\
&= \frac{(2n-p)!}{(2n)!} \frac{(2m-p)!}{(2m)!} \binom{p}{r} [m]_r [m]_{p-r} [n]_r [n]_{p-r} \\
&= \frac{(2n-p)!}{(2n)!} \frac{(2m-p)!}{(2m)!} c_{(m,n)}^{(p,r)}.
\end{aligned} \tag{4.65}$$

These coefficients obey  $c_{(m,n)}^{(p,p-r)} = c_{(m,n)}^{(p,r)}$ . As a consequence, they obey  $\sum_{r=0}^p (-1)^r c_{(m,n)}^{(p,r)} = 0$  for  $p$  odd, and as a result, one obtains

$$w^m \star w^n = \sum_{p=0}^{2m} \left(\frac{1}{2}\right)^p \frac{(2m-p)!}{(2m)!} b_p^{(2m,2n)} \left[ \sum_{r=0}^p (-1)^r c_{(m,n)}^{(p,r)} \right] w^{m+n-p}, \tag{4.66}$$

where only the even values of  $p$  contribute. In the special case  $m = 1$ , we can use (2.45) to evaluate the above expression:

$$w \star w^n = w^{n+1} - \frac{n^2}{4} \frac{(2n-1+\nu)(2n+1-\nu)}{(2n-1)(2n+1)} w^{n-1}, \tag{4.67}$$

which reproduces eq. (3.8) of [47], where this product was considered in the context of fractional spin gravity.

Using  $V_0^0 = J_0 = \frac{1}{2}w$ , we are lead to the identification  $V_0^{n-1} = \left(\frac{1}{2}\right)^n w^n$ . To understand the origin of this dictionary, let us clarify the difference between  $w^n$  and  $\underbrace{w \star \dots \star w}_{n \text{ times}} \equiv w^{\star n}$ . The latter expression can be expanded in a sum of Weyl-ordered monomials in the oscillators  $q_\alpha$ , starting with the maximum degree  $2n$  monomial corresponding to  $w^n$ , together with lower-degree monomials. On the other hand, the generators  $V_m^s$  being part of the enveloping algebra of  $\mathfrak{sl}(2, \mathbb{R})$  can be expressed as polynomials in  $J_0$  and  $J_\pm$ . In the enveloping algebra picture, these generators are defined in terms of nested commutators of  $\mathfrak{sl}(2, \mathbb{R})$  generators:<sup>10</sup>

$$V_m^{s-1} := (-1)^{s-m} \frac{(s+m)!}{(2s)!} \underbrace{[J_-, [J_-, [\dots, [J_-, (J_+)^s] \dots]]}_{s-m \text{ times}} = (-1)^{s-m} \frac{(s+m)!}{(2s)!} (\text{Ad}_{J_-})^{s-m} (J_+)^s \tag{4.68}$$

which, upon using the commutation relations of  $\mathfrak{sl}(2, \mathbb{R})$ , can be reduced to a polynomial in  $J_0, J_\pm$ . This is reminiscent of the fact that  $w^n$  is naturally expressed as a linear combination of “star power”  $w^{\star k}$ ,  $0 \leq k \leq n$ :

$$\begin{aligned}
w^n &= w \star w^{n-1} + \frac{(n-1)^2}{4} \frac{(2n-3+\nu)(2n-1-\nu)}{(2n-3)(2n-1)} w^{n-2} \\
&= w \star \left( w \star w^{n-2} + \frac{(n-2)^2}{4} \frac{(2n-5+\nu)(2n-3-\nu)}{(2n-5)(2n-3)} w^{n-3} \right) \\
&\quad + \frac{(n-1)^2}{4} \frac{(2n-3+\nu)(2n-1-\nu)}{(2n-3)(2n-1)} \left( w \star w^{n-3} + \frac{(n-3)^2}{4} \frac{(2n-7+\nu)(2n-5-\nu)}{(2n-7)(2n-5)} w^{n-4} \right) \\
&= \dots
\end{aligned} \tag{4.69}$$

<sup>10</sup>What we call here  $V_m^{s-2}$  corresponds to  $V_m^s$  in the conventions of [45]. There is therefore a shift of 2 units on the spin.

To sum up,

- in the universal enveloping algebra picture,  $(J_0)^n \equiv \underbrace{J_0 \otimes \cdots \otimes J_0}_{n \text{ times}}$  corresponds to taking  $n$  star product of  $\frac{1}{2}w$ , i.e.  $(J_0)^n = \frac{1}{2^n}w^{\star n}$ ;
- the symbol  $w^n$  corresponds, in the enveloping algebra picture, to the  $n$ th power of the adjoint action of  $J_-$  on the  $n$ th power of  $J_+$ , according to (4.68).

For instance, when  $n = 2$ , we have:

$$(\tfrac{1}{2}w)^2 = (\tfrac{1}{2}w) \star (\tfrac{1}{2}w) + \tfrac{1}{48}(1 + \nu)(3 - \nu) \equiv (\tfrac{1}{2}w) \star (\tfrac{1}{2}w) - \tfrac{1}{3}\mathcal{C}_2[\mathfrak{sl}(2, \mathbb{R})] . \quad (4.70)$$

Using (4.68), we can write:

$$V_0^1 = \tfrac{1}{12} [J_-, [J_-, (J_+)^2]] = (J_0)^2 - \tfrac{1}{3}\mathcal{C}_2[\mathfrak{sl}(2, \mathbb{R})] . \quad (4.71)$$

This reproduces the previous expression of  $w^2$  upon making the identifications  $V_0^1 = \frac{1}{4}w^2$  and  $(J_0)^2 = \frac{1}{4}w \star w$ , which motivates (and justifies) the previously proposed relation  $V_0^{n-1} = (\frac{1}{2})^n w^n$ .

Using this dictionary, we can write:

$$\begin{aligned} w^m \star w^n &= \sum_{p=0}^{2m} \left(\tfrac{1}{2}\right)^{p-1} g_{p-1}^{m-1, n-1}(0, 0; \nu) w^{m+n-p} \\ &= \sum_{p=0}^{2m} \left(\tfrac{1}{2}\right)^p \frac{(2m-p)!}{(2m)!} N_{p-1}^{m-1, n-1} b_p^{(2m, 2n)}(\nu) w^{m+n-p} , \end{aligned} \quad (4.72)$$

which leads us to the following identification:

$$b_p^{(2m, 2n)}(\nu) = \binom{2m}{p} \phi_{p-1}^{m-1, n-1}(\nu) =: \binom{2m}{p} \Phi_p^{(m, n)}(\nu) , \quad (4.73)$$

where we used  $N_{p-1}^{m-1, n-1} = \sum_{r=0}^p (-1)^r c_{(p, r)}^{(m, n)}$ . Let us emphasize that the sum (3.53) initially running over all integer values of  $a$  has been truncated to a finite sum from 0 up to  $2m$  (in the special case of interest to us, i.e. where only generators  $V_0^\ell$  are involved) because  $N_{p-1}^{m-1, n-1}$  vanishes for  $p > 2m$ . The reason behind this is that for  $p > 2m$ , either  $[m]_r$  or  $[m]_{p-r}$  vanishes for  $r = 0, \dots, p$ . Recall that we supposed  $m \leq n$ , but had we supposed the opposite, the same identification would have held and the only change would be that the sum should run from 0 up to  $2n$ .

In the special cases  $p = 1$  and  $p = 2m$ , one can check that the previous identification (4.73) holds<sup>11</sup>, as it reproduces the formulas obtained previously from the deformed star product. Indeed,  $b_1^{(2m, 2n)} = 2m$  as for  $p = 1$  the last argument of the hypergeometric function (in the first row) is zero, and therefore this function is equal to one. In the case  $p = 2m$ , the  ${}_4F_3$  becomes a  ${}_3F_2$  as the last

<sup>11</sup>Notice that it also holds for  $p = 0$ , as the hypergeometric function then reduces to 1.



argument in the first row,  $-(p-1)/2$  is equal to the first one in the second row,  $-m+1/2$ . Then one can use Saalschütz's theorem to evaluate it, which yields:

$${}_3F_2 \left[ \begin{matrix} \frac{\nu}{2}, & 1 - \frac{\nu}{2}, & -m, \\ -n + \frac{1}{2}, & n - m + \frac{3}{2} \end{matrix}; 1 \right] = \frac{(\frac{1-2n-\nu}{2})_m (\frac{\nu-2n-1}{2})_m}{(\frac{1-2n}{2})_m (-\frac{2n+1}{2})_m} = \frac{[\frac{2n-1+\nu}{2}]_m [\frac{2n+1-\nu}{2}]_m}{[\frac{2n-1}{2}]_m [\frac{2n+1}{2}]_m} \quad (4.74)$$

where  $(a)_n$  is the raising Pochhammer symbol, i.e.  $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ , which obey  $(-a)_n = (-1)^n [a]_n$  that we used in the second equality.

In order to prove that the structure constants of the lone star-product (3.54) give the structure constants of  $\mathfrak{hs}[\lambda]$ , we need to show that they obey the same recurrence relation (2.29) as the structure constant for the deformed star-product. In other words, we need to prove that the following relation is true:

$$\begin{aligned} \binom{2m+2}{p} \phi_{p-1}^{(m,n-1)} &= \binom{2m}{p} \phi_{p-1}^{(m-1,n-1)} + 2 \binom{2m}{p-1} \phi_{p-2}^{(m-1,n-1)} \\ &\quad + \frac{(n+m-p+3/2+\nu/2)(n+m-p+5/2-\nu/2)}{(n+m-p+3/2)(n+m-p+5/2)} \binom{2m}{p-2} \phi_{p-3}^{(m-1,n-1)}, \end{aligned} \quad (4.75)$$

which is nothing but the recurrence relation (2.29) that is satisfied by the “even-to-even” coefficients  $b_p^{(2m,2n)}$ .

The relation (4.75) was proven in [36] (that we will not reproduce here), where a different point of view from ours was adopted. In our paper we started by deriving a recurrence relation between the structure constants of the star product between two monomials in the deformed oscillators, i.e. the elements of  $\mathfrak{hs}[\lambda]$ . We were also able to compute explicitly the first and last of these coefficients, and match them with the corresponding  $\mathcal{W}_\infty^{\text{PRS}}$  ones, upon identifying  $w$  with  $\frac{1}{2}J_0$ . Finally, the element of our proof is the validity of (4.75), showing that the structure constants in the bosonic sector of  $\mathfrak{hs}[\lambda]$  are those of the wedge subalgebra of  $\mathcal{W}_\infty^{\text{PRS}}$ . We end up with :

$$b_p^{(2m,2n)} = \binom{2m}{p} {}_4F_3 \left[ \begin{matrix} \frac{\nu}{2}, & 1 - \frac{\nu}{2}, & -\frac{p}{2}, & -\frac{p-1}{2} \\ -m + \frac{1}{2}, & -n + \frac{1}{2}, & n + m - p + \frac{3}{2} \end{matrix}; 1 \right] \equiv \binom{2m}{p} \Phi_p^{(m,n)}(\nu). \quad (4.76)$$

This hypergeometric function is in fact truncated because of the Pochhammer symbol of the negative integer  $-\frac{p}{2}$  or  $-\frac{p-1}{2}$ . It can be rewritten as a finite sum, namely:

$$\begin{aligned} b_p^{(2m,2n)} &= \binom{2m}{p} \sum_{k=0}^{[p/2]} \frac{(\frac{\nu}{2})_k (1 - \frac{\nu}{2})_k (-\frac{p}{2})_k (-\frac{p-1}{2})_k}{k! (\frac{1-2m}{2})_k (\frac{1-2n}{2})_k (n+m-p+3/2)_k} \\ &= \binom{2m}{p} \sum_{k=0}^{[p/2]} \frac{(\frac{\nu}{2})_k (1 - \frac{\nu}{2})_k [\frac{p}{2}]_k [\frac{p-1}{2}]_k}{k! [\frac{2m-1}{2}]_k [\frac{2n-1}{2}]_k (n+m-p+3/2)_k}, \end{aligned} \quad (4.77)$$

where  $[x]$  stands for the integer part of  $x$ . From the identification detailed in a previous paragraph between generators of the wedge subalgebra of  $\mathcal{W}_\infty^{\text{PRS}}$  and monomials of even powers in the deformed oscillators, we can infer that in this case, the structure constants have the same expression whether

the highest monomial is on the left or the right side of the product, i.e.  $\bar{b}_p^{(2n,2m)} = b_p^{(2n,2m)}$ . One way to check this is to look at the recurrence relation linking the coefficients  $\bar{b}_p^{(2n,2m)}$ :

$$\begin{aligned}\bar{b}_p^{(2n,2m+2)} &= \bar{b}_p^{(2n,2m+1)} + \bar{Y}_{2(n+m-p+1)+1} \bar{b}_{p-1}^{(2n,2m+1)} \\ &= \bar{b}_p^{(2n,2m)} + (\bar{Y}_{2(n+m-p+1)+1} + \bar{Y}_{2(n+m-p+1)}) \bar{b}_{p-1}^{(2n,2m)} + \bar{Y}_{2(n+m-p+1)+1} \bar{Y}_{2(n+m-p+2)} \bar{b}_{p-2}^{(2n,2m)} \\ &= \bar{b}_p^{(2n,2m)} + 2\bar{b}_{p-1}^{(2n,2m)} + \frac{(n+m-p+3/2+\nu/2)(n+m-p+5/2-\nu/2)}{(n+m-p+3/2)(n+m-p+5/2)} \bar{b}_{p-2}^{(2n,2m)}, \quad (4.78)\end{aligned}$$

which is indeed the same recurrence relation as the one obeyed by coefficients  $b_p^{(2m,2n)}$ . Added to the fact that the exact expressions we obtained for  $p=1$  and  $p=m$  coincide with those of the  $b$ -type coefficients, it justifies the equality  $\bar{b}_p^{(2n,2m)} = b_p^{(2m,2n)}$ . Therefore, the coefficients entering the star product of two monomials where the one of highest degree is on the left, and an even power of the deformed oscillators, are:

$$\bar{b}_p^{(2n,2m)} = \binom{2m}{p} \Phi_p^{(m,n)}(\nu) = b_p^{(2m,2n)}. \quad (4.79)$$

## 4.2 The $\mathbb{Z}_2$ -graded case of $\mathfrak{shs}[\lambda]$

We can now use the recurrence relation (2.29) to derive the coefficients  $b_p^{(2m+1,2n)}$ :

$$b_p^{(2m+1,2n)} = \pi(b_p^{(2m,2n)}) + \frac{n+m-p+3/2+k\nu/2}{n+m-p+3/2} \pi(b_{p-1}^{(2m,2n)}). \quad (4.80)$$

Finally, we need the coefficients in the case where  $n$  is odd. Using (2.25), one can then show that the following recurrence relation holds:

$$b_p^{(m,n+1)} = \frac{n+1-p}{n+1} b_p^{(m,n)} + \frac{m+1-p}{n+1} \bar{Y}_{n+m-2(p-1)} b_{p-1}^{(m,n)}. \quad (4.81)$$

This enables us to derive the remaining coefficients:

$$b_p^{(2m,2n+1)} = \frac{2n-p+1}{2n+1} b_p^{(2m,2n)} + \frac{2m-p+1}{2n+1} \frac{n+m-p+3/2-k\nu/2}{n+m-p+3/2} b_{p-1}^{(2m,2n)}, \quad (4.82)$$

$$\begin{aligned}b_p^{(2m+1,2n+1)} &= \frac{2n-p+1}{2n+1} \pi(b_p^{(2m,2n)}) + 2 \frac{n+m-p+3/2+k\nu/2}{2n+1} \pi(b_{p-1}^{(2m,2n)}) \\ &\quad + \frac{2m-p+2}{2n+1} \frac{n+m-p+3/2+k\nu/2}{n+m-p+3/2} \frac{n+m-p+5/2+k\nu/2}{n+m-p+5/2} \pi(b_{p-2}^{(2m,2n)}).\end{aligned} \quad (4.83)$$

Turning now to the  $\bar{b}$  coefficients, and using the equality  $\bar{b}_p^{(2n,2m)} = b_p^{(2m,2n)}$ , we can deduce:

$$\bar{b}_p^{(2n,2m+1)} = \bar{b}_p^{(2n,2m)} + \frac{n+m-p+3/2-k\nu/2}{n+m-p+3/2} \bar{b}_{p-1}^{(2n,2m)}. \quad (4.84)$$

Finally, one can show the following other recurrence relation:

$$\bar{b}_p^{(n+1,m)} = \frac{n-p+1}{n+1} \pi(\bar{b}_p^{(n,m)}) + \frac{m-p+1}{n+1} Y_{n+m-2(p-1)}^+ \pi(\bar{b}_{p-1}^{(n,m)}), \quad (4.85)$$

yielding:

$$\bar{b}_p^{(2n+1,2m)} = \frac{2n-p+1}{2n+1} \pi(\bar{b}_p^{(2n,2m)}) + \frac{2m-p+1}{2n+1} \frac{n+m-p+3/2+k\nu/2}{n+m-p+3/2} \pi(b_p^{(2n,2m)}), \quad (4.86)$$

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$$\begin{aligned} \bar{b}_p^{(2m+1,2n+1)} = & \frac{2n-p+1}{2n+1} \pi(\bar{b}_p^{(2m,2n)}) + 2 \frac{n+m-p+3/2+k\nu/2}{n+m-p+3/2} \pi(\bar{b}_{p-1}^{(2m,2n)}) \\ & + \frac{2m-p+2}{2n+1} \frac{n+m-p+3/2+k\nu/2}{n+m-p+3/2} \frac{n+m-p+5/2+k\nu/2}{n+m-p+5/2} \pi(\bar{b}_{p-2}^{(2m,2n)}). \end{aligned} \quad (4.87)$$

Comparing these coefficients to the previous ones (the  $b$ -type ones), one can notice the following relations:

$$\bar{b}_p^{(2n,2m)} = b_p^{(2m,2n)}, \quad \bar{b}_p^{(2n+1,2m+1)} = b_p^{(2m+1,2n+1)}, \quad (4.88)$$

$$\bar{b}_p^{(2n+1,2m)} = \pi(b_p^{(2m,2n+1)}), \quad \bar{b}_p^{(2n,2m+1)} = \pi(b_p^{(2m+1,2n)}) \quad (4.89)$$

**Reconciling barred and unbarred coefficients.** A puzzling feature of the above derivation of the structure constants for the free algebra generated by the deformed oscillator (modulo their commutation relation) is the two types of coefficients that we designated by  $b_p^{(m,n)}$  and  $\bar{b}_p^{(n,m)}$ , distinguishing between two situations: namely whether the lower degree monomial (of deg.  $m$ ) is on the left or the right side of the star product. This distinction arises from the way we defined both kind of structure constants, i.e. with all the lower order monomial in the oscillators on the right side of those coefficients. Having in mind an “operator” form for the star product:

$$f(q) \star g(q) = f(q) K(\overleftarrow{\partial}, \overrightarrow{\partial}, \overleftarrow{\Delta}, \overrightarrow{\Delta}, k\nu) g(q) \quad (4.90)$$

with  $K$  some function of the derivative and homogeneity operators  $\overrightarrow{\Delta} := q^\alpha \frac{\partial}{\partial q^\alpha}$ , as well as of  $k\nu$ , it seems more natural to look at the star product of two monomial in the following form:

$$q_{\alpha(m)} \star q_{\beta(n)} = \sum_{p=0}^{\min(m,n)} q_{\alpha(m-p)} \left( (i\epsilon_{\alpha\beta})^p c_p^{(m,n)} \right) q_{\beta(n-p)}, \quad (4.91)$$

where the  $k$ -dependent structure constants  $c_p^{(m,n)}$  appear in the middle, between two Weyl-ordered monomials.

The monomial  $q_{\alpha(m-p)}$  and  $q_{\beta(n-p)}$  on the right hand side are naturally interpreted as resulting from the action of  $p$  derivatives acting both on the right and on the left. Thus, the above equation can naturally be seen as the expansion of  $K$  in power of the derivative operator, and the coefficients  $c_p^{(m,n)}$  as the (possibly re-summed) action of the homogeneity operators (also carrying various powers of  $k\nu$ ) on the monomials. One could expect the homogeneity operators acting on the left and on the right to be on the same footing, in the same sense as for the derivatives: when expanding  $K$  it appears that each time a derivative acts on the left, another one comes that acts on the right (we know it is true for the standard Moyal star product, and because of its associative nature, we can expect it will remain true for the deformed star product). Let us try to see if at least some of these expectations are realised by relating the coefficients  $b_p^{(m,n)}$  and  $\bar{b}_p^{(m,n)}$ . To do so, we can just rewrite the expansion of  $q_{\alpha(m)} \star q_{\beta(n)}$  as above (hereafter we will assume  $m \leq n$ ):

$$q_{\alpha(m)} \star q_{\beta(n)} = \sum_{p=0}^m \frac{(i\epsilon_{\alpha\beta})^p n!}{(n-p)!} q_{\alpha(m-p)} \pi^{(m-p)}(b_p^{(m,n)}) q_{\beta(n-p)} \quad (4.92)$$


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$$q_{\alpha(n)} \star q_{\beta(m)} = \sum_{p=0}^m \frac{(i\epsilon_{\alpha\beta})^p n!}{(n-p)!} q_{\alpha(n-p)} \pi^{(n-p)}(\bar{b}_p^{(n,m)}) q_{\beta(m-p)} . \quad (4.93)$$

Before comparing  $\bar{c}_p^{(m,n)} := \pi^{(m-p)}(b_p^{(m,n)})$  with  $\tilde{c}_p^{(n,m)} := \pi^{(n-p)}(\bar{b}_p^{(n,m)})$ , let us prove the useful identity:

$$Y_{n+i}^{(-)^i} = \bar{Y}_{n+i}^{(-)^{n-1}}, \quad (4.94)$$

where we introduced the notation  $\bar{Y}_n^{(-)^\ell} = 1 + (-1)^\ell k\nu \left[ \frac{P_n}{n} - \frac{P_{n+1}}{n+1} \right]$ .

*Proof.* We can distinguish two different cases:

- $n+i$  is even, which implies  $\bar{Y}_{n+i}^{(-)^{n-1}} = 1 + (-1)^n k\nu \frac{1}{n+i+1} = 1 + (-1)^i k\nu \frac{1}{n+i+1}$  ( $n+i$  being even,  $n$  and  $i$  have the same parity);
- $n+i$  odd, which implies  $\bar{Y}_{n+i}^{(-)^{n-1}} = 1 + (-1)^{n-1} k\nu \frac{1}{n+i} = 1 + (-1)^i k\nu \frac{1}{n+i}$  ( $n+i$  being odd,  $n$  and  $i$  have opposed parity).

Therefore we have

$$\bar{Y}_{n+i}^{(-)^{n-1}} = Y_{n+i}^{(-)^i} . \quad (4.95)$$

□

Using this last identity, we can easily show that  $\bar{c}_p^{(m,n)} = \tilde{c}_p^{(n,m)} \equiv c_p^{(m,n)}$ .

*Proof.* We just have to apply the sign flips encoded in the action of  $\pi$ :

$$\begin{aligned} c_p^{(m,n)} &= \pi^{(m-p)}(b_p^{(m,n)}) = \prod_{\ell=0}^{p-1} \sum_{i_\ell=i_{\ell-1}}^{m-p} Y_{n+i_\ell-\ell}^{(-)^{i_\ell+\ell-(p-1)}} \\ &= \prod_{\ell=0}^{p-1} \sum_{i_\ell=i_{\ell-1}}^{m-p} \bar{Y}_{n+i_\ell-\ell}^{(-)^{n-p}} = \pi^{(n-p)}(\bar{b}_p^{(n,m)}) = \tilde{c}_p^{(n,m)} . \end{aligned} \quad (4.96)$$

□

This enables us to “reconcile” both types of structure constants  $b$  and  $\bar{b}$  in only one:

$$c_p^{(m,n)} = \prod_{\ell=0}^{p-1} \sum_{i_\ell=i_{\ell-1}}^{\min(m,n)-p} Y_{\max(m,n)+i_\ell-\ell}^{(-)^{i_\ell-\ell+p-1}} , \quad (4.97)$$

the remaining difference of status between the lower and the higher degree of the two monomials being (a priori) due to the action of the homogeneity operators that probably enters the above formula in a re-summed form.

### 4.3 Fourier mode basis of $\mathfrak{shs}[\lambda]$ and supertrace

It is common to present the  $\mathcal{N} = 2$  super- $\mathcal{W}_\infty$  algebra, of which  $\mathfrak{shs}[\lambda]$  is the wedge subalgebra, in a way that makes the  $\mathcal{N} = 2$  super-multiplet structure explicit (e.g. [48, 49]). The content of  $\mathfrak{shs}[\lambda]$  in terms of super-multiplet is as follows:

$$\left\{ \begin{array}{cc} 1 & \\ 3/2 & 3/2 \\ 2 & \end{array} \right\}, \quad \left\{ \begin{array}{cc} 2 & \\ 5/2 & 5/2 \\ 3 & \end{array} \right\}, \quad \left\{ \begin{array}{cc} 3 & \\ 7/2 & 7/2 \\ 4 & \end{array} \right\}, \dots \quad (4.98)$$

with each super-multiplet grouping generators of spins  $s, s + 1/2$  and  $s + 1$ , the spin-1 generator corresponding to  $J$ , see Section 1. In this section, we give the precise dictionary between  $\mathfrak{shs}[\lambda]$  generators in the Fourier mode basis (familiar in the CFT context) and in terms of the deformed oscillators. Generically, the spin  $s$  generator is realised as a monomial of order  $\ell = 2(s - 1)$  in the deformed oscillators, and there are  $2s$  modes for this generator, labelled by  $|m| \leq s - 1$ , such that the difference between the number of the two different kind of oscillators ( $q_1$  and  $q_2$ ) is  $2m$ ; see e.g. [49].

**Bosonic sector.** We have established that

$$V_0^s = \frac{1}{2^{s+1}} w^{s+1}. \quad (4.99)$$

By rescaling the generators  $V_m^s$  in order to reintroduce the scaling parameter  $q$  (that we set earlier to  $1/4$  in order to make contact with [47]), renamed hereafter  $\gamma$  so as to avoid any confusion with the deformed oscillators, one can check, using (4.62), that the generators

$$V_m^s = 2^{s-1} \gamma^s (a^+)^{s+1-m} (a^-)^{s+1+m} \quad (4.100)$$

indeed verify the  $\mathcal{W}_\infty^{\text{PRS}}$  commutation relations:

$$\left[ V_m^s, V_n^{s'} \right]_\star = \sum_{p=0}^{\min(s, s')} \gamma^{2p} g_{2p}^{s, s'}(m, n; \nu) V_{m+n}^{s+s'-2p}, \quad (4.101)$$

where  $g_{2p}^{s, s'}(m, n; \nu)$  is given by (3.54), (3.55) and (3.56).

In order to present  $\mathfrak{shs}[\lambda]$  in the way described above, we need to redefine its generators in the Fourier mode basis as follows:

$$T_m^{(s)\sigma} := \frac{1}{2} (2\gamma)^{s-2} (a^+)^{s-1-m} (a^-)^{s-1+m} P_\sigma, \quad P_\sigma := \frac{1+\sigma k}{2}, \quad s \geq 2, \quad |m| \leq s-1. \quad (4.102)$$

Splitting them into bosonic generators  $V_m^{(s)\sigma}$  related to the ones introduced in (4.100) by  $s \rightarrow s-2$ , and the fermionic ones that we will rewrite:

$$T_{m'}^{(s)\sigma} \equiv G_{m, \epsilon}^{(s)\sigma} := \frac{1}{2} (2\gamma)^{s-2} (a^+)^{s-1-(m-\epsilon/2)} (a^-)^{s-1+m-\epsilon/2} P_\sigma, \quad (4.103)$$

where we introduced the split  $m' = m - \epsilon/2$  as for fermionic generators, both  $s$  and  $m'$  are half integer. The maximal finite-dimensional subsuperalgebra contained in  $\mathfrak{shs}[\lambda]$  is  $\mathfrak{osp}(2|2)$  and is spanned by

the generators  $T_m^{(s),\pm}$  in the sector where  $s \in \{1, 3/2, 2\}$ . Explicitly, in terms of the generators  $\{Q_m^{(i)}\}$ ,  $m = \pm 1/2$ , together with  $\{L_{-1}, L_0, L_1\}$  and  $J$ , where we choose  $\gamma = 1/2$  and

$$Q_m^{(i)} := \frac{1}{2} (u^{+\alpha})^{1/2-m} (u^{-\alpha})^{\frac{1}{2}+m} Q_\alpha^{(i)}, \quad m \in \{\frac{1}{2}, -\frac{1}{2}\}, \quad (4.104)$$

$$L_m := i(u^{+\alpha})^{1-m} (u^{-\alpha})^{1+m} T_{\alpha\alpha}, \quad m \in \{-1, 0, 1\}. \quad (4.105)$$

By comparison with the presentation (1.8), we have

$$\{Q_m^{(i)}, Q_n^{(j)}\} = \sigma_3^{ij} L_{m+n} + \frac{m-n}{2} \tau^{ij} J, \quad [L_m, Q_n^{(i)}] = \frac{1}{2} \left(\frac{m}{2} + n\right) Q_{n-m}^{(i)}, \quad [J, Q_m^{(i)}] = \tau^{ij} Q_m^{(j)}. \quad (4.106)$$

**Fermionic sector.** Using the previous definitions, we arrive at

$$V_m^{(s),\sigma} \star G_{n,\epsilon}^{(s'),\sigma'} = \frac{1}{2} \sum_{p=0}^{\min(2s-2, 2s'-1)} \frac{1}{2p!} \gamma^{p-1} \tilde{N}_{p,\epsilon}^{s,s'}(m, n) \varphi_p^{(s,s')}(\nu) G_{n,\epsilon}^{(s+s'-[p+1]),\sigma'} \delta_{\sigma+\sigma',0}, \quad (4.107)$$

with

$$\tilde{N}_{p,\epsilon}^{s,s'}(m, n) = \sum_{r=0}^p (-1)^r \binom{p}{r} [s-1-m]_r [s-1+m]_{p-r} [s'-1-n+\frac{\epsilon}{2}]_{p-r} [s'-1+n-\frac{\epsilon}{2}]_r \quad (4.108)$$

and

$$\varphi_p^{(s,s')}(\nu) = \frac{1}{2(s'-1)} \left( (2s - (p+1)) \pi(\Phi_p^{(s-1, s'-3/2)}) + p \frac{s+s'-1-p+k\nu/2}{s+s'-1-p} \pi(\Phi_{p-1}^{(s-1, s'-3/2)}) \right). \quad (4.109)$$

Using (4.88), one can show:

$$\left[ V_m^{(s),\sigma}, G_{n,\epsilon}^{(s'),\sigma'} \right]_\star = \sum_{p=0}^{2(\min(s,s')-1)} \frac{1}{4p!} \gamma^{p-1} \tilde{N}_{p,\epsilon}^{s,s'}(m, n) \left( \varphi_p^{(s,s')} - (-1)^p \pi(\varphi_p^{(s,s')}) \right) G_{n,\epsilon}^{(s+s'-[p+1]),\sigma'} \delta_{\sigma'+\sigma,0}. \quad (4.110)$$

One the other hand,

$$G_{m,\epsilon}^{(s),\sigma} \star G_{n,\epsilon'}^{(s'),\sigma'} = \sum_{p=0}^{2\min(s,s')-1} \frac{1}{2p!} \gamma^p M_{p,(\epsilon,\epsilon')}^{s,s'}(m, n) \psi_p^{(s,s')}(\nu) V_{m+n-(\epsilon+\epsilon')/2}^{(s+s'-p),\sigma'} \delta_{\sigma+\sigma',0}, \quad (4.111)$$

with

$$M_{p,(\epsilon,\epsilon')}^{s,s'}(m, n) = \sum_{r=0}^p (-1)^r \binom{p}{r} [s-1-m+\frac{\epsilon}{2}]_r [s-1+m-\frac{\epsilon}{2}]_{p-r} [s'-1-n+\frac{\epsilon'}{2}]_{p-r} [s'-1+n-\frac{\epsilon'}{2}]_r \quad (4.112)$$

and

$$\begin{aligned} \psi_p^{(s,s')} &= \frac{(2(s-1)-p)(2(s'-1)-p)}{4(s-1)(s'-1)} \pi(\Phi_p^{(s-3/2, s'-3/2)}) \\ &+ 2p \frac{(s+s'-p-3/2+k\nu/2)}{4(s-1)(s'-1)} \pi(\Phi_{p-1}^{(s-3/2, s'-3/2)}) \\ &+ \frac{p(p-1)}{4(s-1)(s'-1)} \frac{(s+s'-3/2-p+k\nu/2)(s+s'-1/2-p+k\nu/2)}{(s+s'-3/2-p)(s+s'-1/2-p)} \pi(\Phi_{p-2}^{(s-3/2, s'-3/2)}). \end{aligned} \quad (4.113)$$

Using  $M_{p,(\epsilon,\epsilon')}^{s,s'}(m, n) = (-1)^p M_{p,(\epsilon',\epsilon)}^{s',s}(n, m)$ , one ends up with:

$$\left\{ G_{m,\epsilon}^{(s),\sigma}, G_{n,\epsilon'}^{(s'),\sigma'} \right\}_\star = \sum_{p=0}^{2(\min(s,s')-1)} \frac{1}{p!} \gamma^p M_{p,(\epsilon,\epsilon')}^{s,s'}(m, n) \psi_p^{(s,s')}(\nu) \left( V_{m+n-(\epsilon+\epsilon')/2}^{(s+s'-p),\sigma'} - (-1)^p V_{m+n-(\epsilon+\epsilon')/2}^{(s+s'-p),-\sigma'} \right) \delta_{\sigma+\sigma',0} \quad (4.114)$$

**Supertrace.** Turning to the supertrace, it is easy to extract the maximal contraction of the star-product of two monomials of same degree  $n$ , thereby reproducing the formula given in [4]. Denoting  $F[f(q, k)] := f(0, k)$ , one directly obtains

$$F[(q_\alpha)^n \star (q_\beta)^m] = \delta_{n,m} \epsilon_{\alpha_1 \beta_1} \dots \epsilon_{\alpha_n \beta_n} T_n(k, \nu), \quad (4.115)$$

where

$$\begin{aligned} T_{2m+2}(k, \nu) &= (-1)^{m+1} (2m+2)! \left(1 - \frac{k\nu}{2m+3}\right) (1+k\nu) \prod_{\ell=1}^m \left(1 - \frac{\nu^2}{(2\ell+1)^2}\right), \\ T_{2m+1}(k, \nu) &= i(-1)^m (2m+1)! (1+k\nu) \prod_{\ell=1}^m \left(1 - \frac{\nu^2}{(2\ell+1)^2}\right). \end{aligned} \quad (4.116)$$

This result agrees with  $T_n(k, \nu) = i^n n! b_n^{(n,n)} = i^n n! \bar{b}_n^{(n,n)}$ , as these coefficients are the same for  $n$  and  $m$  both even or odd. Indeed:

$$\prod_{\ell=1}^m \left(1 - \frac{\nu^2}{(2\ell+1)^2}\right) = \prod_{\ell=1}^m \frac{(\ell + \frac{1+\nu}{2})(\ell + \frac{1-\nu}{2})}{(\ell + \frac{1}{2})^2} = \frac{(\frac{3-\nu}{2})_m (\frac{3+\nu}{2})_m}{((\frac{3}{2})_m)^2}, \quad (4.117)$$

$$b_{2m+1}^{(2m+1, 2m+1)} = \frac{(\frac{1+\nu}{2})_{m+1} (\frac{3-\nu}{2})_m}{(\frac{1}{2})_{m+1} (\frac{3}{2})_m} = (1+\nu) \frac{(\frac{3+\nu}{2})_m (\frac{3-\nu}{2})_m}{((\frac{3}{2})_m)^2}, \quad b_{2m}^{(2m, 2m)} = \frac{(\frac{1+\nu}{2})_m (\frac{3-\nu}{2})_m}{(\frac{1}{2})_m (\frac{3}{2})_m} \quad (4.118)$$

$$\Rightarrow T_n(k, \nu) = i^n n! b_n^{(n,n)}. \quad (4.119)$$

The supertrace  $\text{Str}_\nu[f(q, k)] = f(0, -\nu)$  can readily be obtained from this result. What is obvious from this formula is that fact that the supertrace degenerates for critical values of  $|\nu|$ .

Another special case (referred to as “hypercritical” in [47]) is  $|\nu| = 1$ : because of the factor  $(1+k\nu)$  (upon setting  $k = \pm 1$ ), the supertrace degenerates and becomes identically zero. This is a well known feature of  $\mathfrak{hs}[\lambda]$  at  $\lambda = \pm 1$ . In this case,  $\mu = 0$  which means that  $\mathfrak{hs}[\lambda]$ , as an associative algebra, is the universal enveloping algebra of  $\mathfrak{sl}(2, \mathbb{R})$  quotiented by  $\langle \mathcal{C}_2 \rangle$ , i.e. where the identity operator is removed from the UEA. As a result, the invariant bilinear trace being defined as taking the identity component of the product of two elements, is degenerate. To circumvent this problem, one usually rescales the trace by  $\frac{1}{\lambda^2-1}$ , or equivalently in our case, by  $\frac{1}{1\pm\nu}$ .

## 5 Conclusion

Using the realisation of  $\mathfrak{shs}[\lambda]$  provided by the associative algebra made out of all *symmetrised* (even and odd) powers of the deformed oscillators endowed with the star-commutator defined by (1.10), we have given closed-form formulae for the structure constants of  $\mathfrak{shs}[\lambda]$ . In particular, in the bosonic case we gave a formal proof that the structure constants postulated in [31] are indeed those of  $\mathfrak{hs}[\lambda]$ , thereby completing the work of [36]. Our proof relies on the associativity of the deformed star product, from which follows the recurrence relation (2.29) (resp. (2.48)) linking the structure constants  $b_p^{(m,n)}$  (resp.  $\bar{b}_p^{(m,n)}$ ) involved in the product of two monomials of degree  $m$  and  $n$  (resp.  $n$  and  $m$ ) to those for

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two monomials of degree  $m+1$  and  $n$  (resp.  $n$  and  $m+1$ ). We were able to give closed-form formulae, (2.31) and (2.49), for the structure constants verifying the aforementioned recurrence relation. These formulae are expressed in terms of nested sums of products of some elementary building blocks, the linear functions denoted  $Y_n^\pm$  and  $\bar{Y}_n$ , see (2.23) and (2.25). The latter functions, via their dependence in  $\nu$ , encode the deformation appearing in the star product of a single oscillator with an arbitrary Weyl-ordered monomial in the oscillators when one replaces the non-deformed oscillators (with  $\nu = 0$ ) by the deformed ones. Finally, we were able to show, in the bosonic case, that:

- (1) our closed formula agrees with the structure constants  $g_p^{m,n}(\nu)$  of [31] for the first two and the last terms ( $b_0^{(m,n)}$ ,  $b_1^{(m,n)}$  and  $b_m^{(m,n)}$ ) appearing in the expansion of the star product in terms of pointwise products of lowest degree monomials;
- (2) the structure constants involved in the star product of even monomials  $b_p^{(m,n)}$  and  $\bar{b}_p^{(n,m)}$ , and the structure constants  $g_p^{m,n}(\nu)$ , verify the same recurrence relation (using a result of [36]).

What the recurrence relations (2.29) and (2.48) show is that knowing the “boundary data”

$$\left\{ b_0^{(m,n)}, b_m^{(m,n)}, \forall m \leq n \in \mathbb{N} \right\} \quad (5.120)$$

is sufficient to reconstruct any of the structure constants. As those of [31] verify the two conditions enumerated above, they therefore are the unique solution of (2.29) and (2.48), and as a consequence, the Lone Star product constructed in [31] is the deformed star product (1.10).

## Acknowledgments

We want to thank Fabien Buisseret for collaboration at the beginning of the project. It is a pleasure to thank Andrea Campoleoni, Shouvik Datta and Tomáš Procházka for discussions on  $\mathcal{W}$  algebras and asymptotic symmetries, as well as Slava Didenko, Zhenya Skvortsov, Philippe Spindel, Per Sundell and Mauricio Valenzuela for discussions on the deformed star product. T.B. also thanks Kevin Morand for various discussions on the construction of  $\mathfrak{hs}[\lambda]$  from deformed oscillators. T.B. is supported by a joint grant “50/50” Université François Rabelais Tours – Région Centre / UMONS.



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